Guaranteed Bounds for Posterior Inference in Universal Probabilistic Programming

Raven Beutner[1] Luke Ong[2] **Fabian Zaiser**[2]

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Our work

- Probabilistic programming: Bayesian statistical models as programs
- Vision: Bayesian inference algorithms for any program

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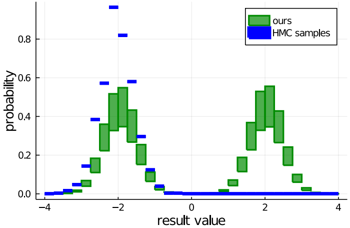
Our contribution:

guaranteed bounds on the posterior

✓ can find errors in inference results

✓ applicable to a broad class of probabilistic programs



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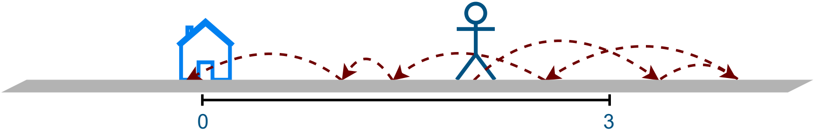


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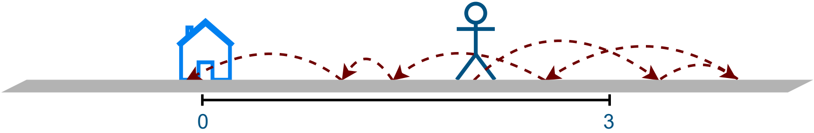


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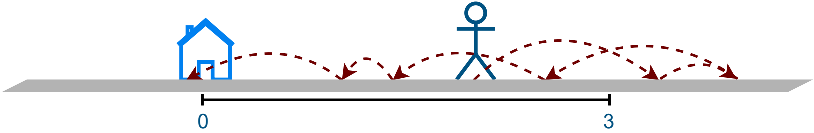


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Posterior distribution $p(start \mid \text{observation})$?

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1. Approximate: posterior $\approx X$

- Monte Carlo (particle filter, MCMC)
- or optimization-based (variational inference)

Stan *Pyro* *Anglican*

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2. Exact: posterior $= X$

- symbolic expression

PSI SOLVER HAKARU ΣΠPL

SPPL

Issues with existing methods

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- **approximate methods:** implicit assumptions, slow convergence

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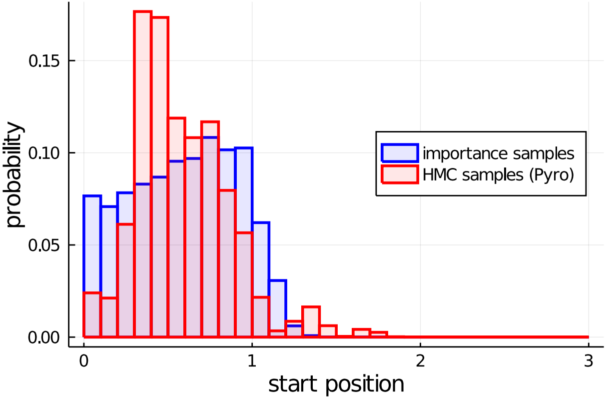

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- to debug approximate inference

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1. interval traces & interval arithmetic (basic idea)
2. interval type system (overapproximation)
3. symbolic execution (optimization of special case)

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Semantics of a probabilistic program

- trace s records **sample**d values, e.g. $\langle 0.23, 0.79 \rangle$
- result value $\mathsf{resval}(s)$ for trace s
- weight $\mathsf{weight}(s)$: product of likelihoods of observations

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Unnormalized posterior of E (*joint probability*):

$$[\![P]\!](E) := \int_{\{\boldsymbol{s} \mid \mathsf{resval}(\boldsymbol{s}) \in E\}} \mathsf{weight}(\boldsymbol{s})\, \mathrm{d}\boldsymbol{s} = \text{“}\,\mathbb{P}(\texttt{start} \in E, \mathsf{obs})\,\text{”}$$

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Cover $\{\boldsymbol{s} \mid \mathsf{resval}(\boldsymbol{s}) \in E\}$ with interval traces $\mathcal{T}$

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$$[\![P]\!](E) \leq \sum_{\boldsymbol{t} \in \mathcal{T}} (\max \mathsf{weight}(\boldsymbol{t})) \, \mathrm{vol}(\boldsymbol{t})$$

Interval trace semantics

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	standard	interval semantics
`start`		
`position`		
`distance`		
trace s	$\langle 0.6, 0.2, -0.8 \rangle$	$\langle [0.5, 0.6], [0.1, 0.2], [-0.9, -0.8] \rangle$
weight weight(s)	1	$[1, 1]$
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Theoretical results

Soundness

For a non-overlapping and exhaustive set of interval traces $\mathcal{T}$:

$$\mathsf{lowerBd}_P^{\mathcal{T}} \leq [\![P]\!] \leq \mathsf{upperBd}_P^{\mathcal{T}}.$$

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Completeness

For all intervals I and $\epsilon > 0$, there is a countable set $\mathcal{T}$ of interval traces (non-overlapping and exhaustive) s.t.

$$\mathsf{upperBd}_P^{\mathcal{T}}(I) - \epsilon \leq [\![P]\!](I) \leq \mathsf{lowerBd}_P^{\mathcal{T}}(I) + \epsilon$$

under mild assumptions about the program P.

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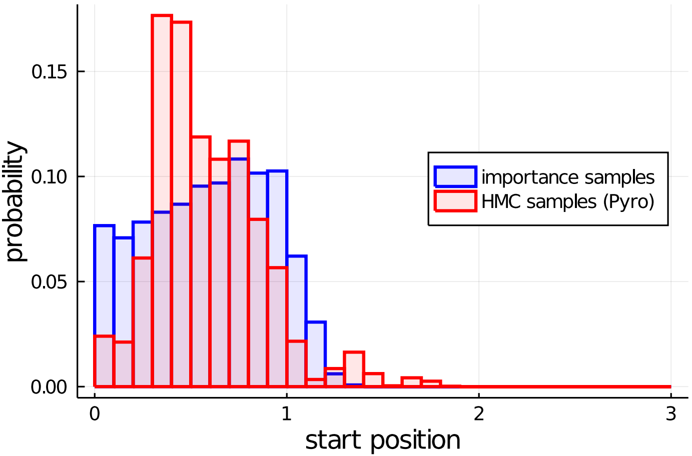

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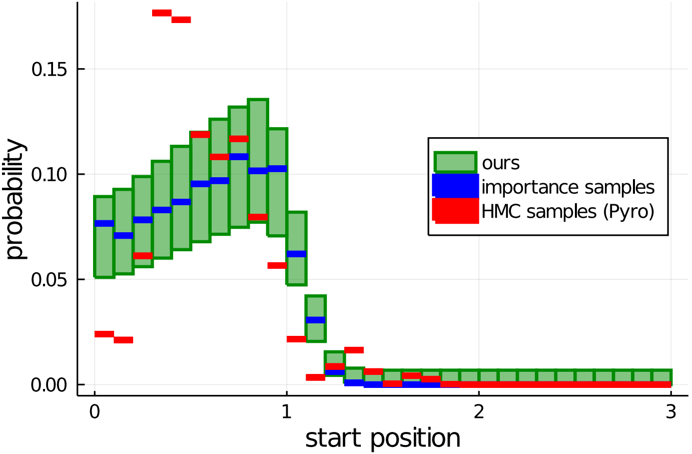
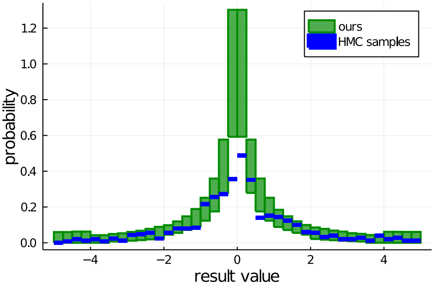
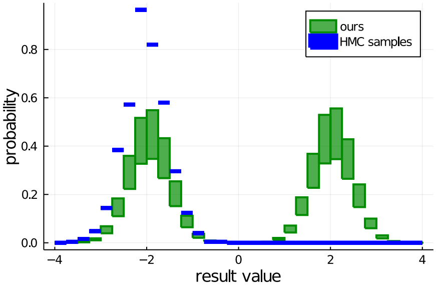


Examples that are hard for MCMC



Neal’s funnel



Mixture model

Comparison with previous work

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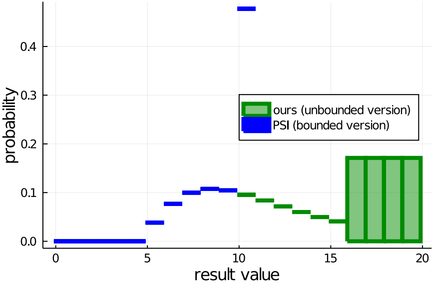
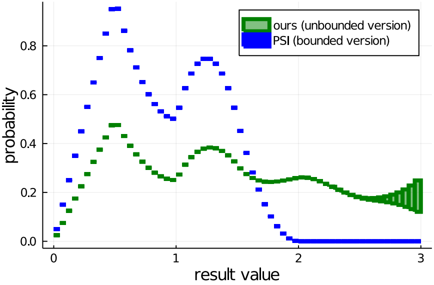
PSI solver (CAV2016)

- consistency check: benchmarks from the PSI repository
- we can handle unbounded loops, contrary to PSI





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- **Comparison with statistical validation methods:** simulation-based calibration

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Limitations

- lots of branching
- high-dimensional models (many `sample`s)

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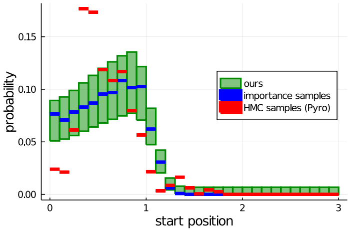
…are a **middle ground** between *approximate* and *exact*:

- guaranteed correct (vs. approximate inference)
- supports many language features (vs. exact inference)

Theory: soundness & completeness

Practice:

- detect issues with inference results
- competitive on existing benchmarks
- guaranteed results for programs that other tools cannot handle



Backup slides

Trace partitioning heuristics

Option 1: split equidistantly in each dimension

Option 2:

- start with the full interval trace $\langle[-\infty, \infty], \dots\rangle$
- pick the next interval $\boldsymbol{t}$ trace or, depending on the input program, select it with a mix of the following criteria
  - high weight $\mathsf{weight}^{\mathbb{I}}(\boldsymbol{t})$
  - wide value interval $\mathsf{resval}^{\mathbb{I}}(\boldsymbol{t})$
  - large volume $\mathrm{vol}(\boldsymbol{t})$
- split that box in half along the dimension that reduces the width of the interval of the posterior expected value the most
- repeat.

Method 1: Interval traces

	standard semantics
traces	$\boldsymbol{s} = \langle 0.2, 0.8 \rangle$
value	$\mathsf{resval}(\boldsymbol{s}) \in \mathbb{R}$
weight	$\mathsf{weight}(\boldsymbol{s}) \in [0, \infty)$
posterior	$[\![P]\!](E)$
	integral over traces $\boldsymbol{s}$

Method 1: Interval traces

Idea: summarize traces using intervals

	standard semantics	interval semantics
traces	$\boldsymbol{s} = \langle 0.2, 0.8 \rangle$	$\boldsymbol{t} = \langle [0.2, 0.3], [0.7, 0.8] \rangle$
value	$\mathsf{resval}(\boldsymbol{s}) \in \mathbb{R}$	$\mathsf{resval}^{\mathbb{I}}(\boldsymbol{t}) \in \mathbb{I}$
weight	$\mathsf{weight}(\boldsymbol{s}) \in [0, \infty)$	$\mathsf{weight}^{\mathbb{I}}(\boldsymbol{t}) \in \mathbb{I}_{[0,\infty)}$
posterior	$[\![P]\!](E)$	$[\mathsf{lowerBd}_P^{\mathcal{T}}(E), \mathsf{upperBd}_P^{\mathcal{T}}(E)]$
	integral over traces $\boldsymbol{s}$	sum over interval traces $\boldsymbol{t}$

Bounding the posterior

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... if $\mathcal{T}$ is a set of interval traces that is "exhaustive" (covers every trace) and where

$$\mathrm{vol}(\langle [a_1, b_1], \ldots, [a_n, b_n] \rangle) := (b_1 - a_1) \times \cdots \times (b_n - a_n)$$

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…if $\mathcal{T}$ is a set of interval traces that is “non-overlapping” and “exhaustive” (covers every trace) and where

$$\mathrm{vol}(\langle[a_1, b_1], \dots, [a_n, b_n]\rangle) := (b_1 - a_1) \times \cdots \times (b_n - a_n)$$

Theoretical results

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$$\text{upperBd}_P^{\mathcal{T}}(I) - \epsilon \leq [\![P]\!](I) \leq \text{lowerBd}_P^{\mathcal{T}}(I) + \epsilon$$

under the assumptions:

- the primitive functions are continuous*
- each **`sample`**d value is used at most once in each condition, **`observe`** statement, and in the return value.

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For all intervals I and $\epsilon > 0$, there is a finite set $\mathcal{T}$ of interval traces (non-overlapping and exhaustive) s.t.

$$[\![P]\!](I) \leq \mathsf{lowerBd}_P^{\mathcal{T}}(I) + \epsilon$$

under the assumptions:

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Method 2: Interval type system

$\rightarrow$ to overapproximate recursion and conditionals (not resolvable by intervals)

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- types keep track of the value and weight interval
- $\vdash P : \left\{ \begin{matrix} [v, v'] \\ [w, w'] \end{matrix} \right\}$ means $\mathsf{resval}(\boldsymbol{s}) \in [v, v']$ and $\mathsf{weight}(\boldsymbol{s}) \in [w, w']$.
- efficient type inference
- uses interval arithmetic & widening to approximate fixpoints

Interval type system

Types:

- unweighted: $\sigma ::= [v, v'] \mid \sigma \to \mathcal{A}$
- weighted: $\mathcal{A} ::= \left\{ \begin{matrix} \sigma \\ [w, w'] \end{matrix} \right\}$

Selected typing rules:

$$\frac{\Gamma; \varphi : \sigma \to \mathcal{A}; x : \sigma \vdash M : \mathcal{A}}{\Gamma \vdash \mu_x^{\varphi}.\, M : \left\{ \begin{matrix} \sigma \to \mathcal{A} \\ [1, 1] \end{matrix} \right\}}$$

$$\frac{\Gamma \vdash M : \left\{ \begin{matrix} \sigma_1 \to \left\{ \begin{matrix} \sigma_2 \\ [e, f] \end{matrix} \right\} \\ [a, b] \end{matrix} \right\} \qquad \Gamma \vdash N : \left\{ \begin{matrix} \sigma_1 \\ [c, d] \end{matrix} \right\}}{\Gamma \vdash M N : \left\{ \begin{matrix} \sigma_2 \\ [a, b] \times^{\mathbb{I}} [c, d] \times^{\mathbb{I}} [e, f] \end{matrix} \right\}}$$

Method 3: Symbolic execution

$\rightarrow$ optimization for a common special case

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For each program path,

- α_k: the k-th **`sample`**
- $\mathcal{V}$: result value, e.g. $\alpha_1 + 2\alpha_2$
- Δ: guards, e.g. $\{\alpha_1 \leq 0, \alpha_1 + \alpha_2 > 1\}$
- Ξ: weights, e.g. $\{\mathrm{pdf}_{\mathrm{Normal}(0,1)}(\alpha_1 - \alpha_2), \mathrm{pdf}_{\mathrm{Normal}(1,2)}(\alpha_3)\}$

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$$[\![P]\!](I) = \sum_{\text{paths}} \int_{\Delta \cup \{\mathcal{V} \in I\}} \left(\prod \Xi\right) \mathrm{d}\alpha$$

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If Δ and $\mathcal{V}$ are affine then use

- polytope volume computation ($\rightarrow$ Vinci tool)
- linear optimization & interval arithmetic

Future work

- better heuristics for finding a "good" set of interval traces $\mathcal{T}$
- integration into an approximate inference algorithm?